Complete chaotic synchronization in mutually coupled time-delay systems

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(Received 28 September 2006; published 2 February 2007)

Complete chaotic synchronization of end lasers has been observed in a line of mutually coupled, time-delayed system of three lasers, with no direct communication between the end lasers. The present paper uses ideas from generalized synchronization to explain the complete synchronization in the presence of long coupling delays, applied to a model of mutually coupled semiconductor lasers in a line. These ideas significantly simplify the analysis by casting the stability in terms of the local dynamics of each laser. The variational equations near the synchronization manifold are analyzed, and used to derive the synchronization condition that is a function of parameters. The results explain and predict the dependence of synchronization on various parameters, such as time delays, strength of coupling and dissipation. The ideas can be applied to understand complete synchronization in other chaotic systems with coupling delays and no direct communication between synchronized subsystems.

I. INTRODUCTION

Synchronized chaotic oscillations have been found in many nonlinear systems, from lasers [1] to neural networks [2]. Many types of synchronization have been observed, including complete synchronization, phase-locking, and generalized synchronization, in the case of unidirectionally coupled systems. For an overview of the large body of work done on synchronization, and its subclasses see, for example, Refs. [3,4]. While extensive work has been done on mutually coupled systems, general analytic techniques for analyzing chaotic synchronization in time-delay mutually coupled systems have not been well developed. Essentially, there exist two methods for analyzing synchronization in coupled systems: A Lyapunov function approach [5] and a master stability approach [6]. Time delays considerably complicate the analysis, possibly introducing infinite degrees of freedom, and resulting in new types of dynamics [5]. The present paper proposes an approach for understanding and predicting chaotic synchronization of time-delayed mutually coupled systems, possessing internal symmetry.

In the case of internal symmetry in a system, where the equations of motion are invariant with respect to interchange of some variables, there is a solution where these variables are exactly equal [7]. For example, for the case of three nonlinear oscillators coupled in a line, if the equations of the outside oscillators are identical, then given the same initial conditions, these oscillators will have identical dynamics, including the possibility of chaotic solutions. In that case, the system can be reduced to two coupled oscillators. If the symmetric solution is asymptotically stable to perturbations off the synchronization manifold, then the dynamics of outside oscillators are synchronized. Thus the requirement for synchronization in the long time limit is that the largest Lyapunov exponent, with respect to perturbations transverse

to the synchronization manifold, is negative, resulting in decay of the initial perturbation back to synchronized state [6]. In general, Lyapunov exponents must be calculated numerically. However, as will be shown, an analytic estimate can be made in some cases by linearizing about the synchronous state.

Although there is extensive work on synchronization of coupled systems, studies of chaotic synchronization in time-delayed systems is much less extensive by comparison. Some analysis exists on synchronization of coupled semiconductor lasers without delays [7,8]. However, it remains to be explained, for example, why in a coupled three laser system, outer lasers show complete synchronization, in the presence of long time delays (compared to the internal dynamics of each laser) and no communication except via the middle laser, which itself is not synchronized with the end lasers [9]. (See Fig. 2 for an example.) The present paper aims to explain this phenomena observed in lasers and other time-delayed systems using ideas from generalized synchronization. The paper is organized as follows: In Sec. II, the general equations for the three subsystems coupled by nearest neighbor interactions with delays are introduced, and the equations are linearized close to the synchronization manifold, using internal symmetry. Section III uses synchronization ideas developed in Sec. II to explain complete synchronization of semiconductor lasers in the presence of long delays and with no direct communication between the outer lasers. Section IV concludes and summarizes.

II. LONG DELAYS AND GENERALIZED SYNCHRONIZATION

We deal with mutually coupled, oscillatory time-delay systems possessing internal symmetry, with respect to interchange of some variables. The system can be broken up into three coupled parts as depicted in Fig. 1: the "center" and

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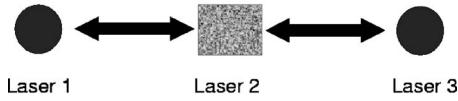*Electronic address: alandsma@cantor.nrl.navy.mil

FIG. 1. A schematic showing how three lasers are coupled in a line. The outer two lasers (circles) are identical, while the middle laser (square) is detuned from the rest.

two identical subparts, that possess symmetry with respect to interchange of variables:

$$\frac{d\mathbf{z_1}}{dt} = \mathbf{F}(\mathbf{z_1}(t)) + \delta_1 \mathbf{G}(\mathbf{z_2}(t - \tau)), \qquad (1)$$

$$\frac{d\mathbf{z_2}}{dt} = \widetilde{\mathbf{F}}(\mathbf{z_2}(t)) + \delta_2 \widetilde{\mathbf{G}}(\mathbf{z_1}(t - \tau), \mathbf{z_3}(t - \tau)), \qquad (2)$$

$$\frac{d\mathbf{z_3}}{dt} = \mathbf{F}(\mathbf{z_3}(t)) + \delta_1 \mathbf{G}(\mathbf{z_2}(t - \tau)), \qquad (3)$$

where $\mathbf{z_i}$ are vector variables of some dimension M and N, for the outer and the middle subsystem, respectively. Variables $\mathbf{z_1}$ and $\mathbf{z_3}$ are symmetric with respect to interchange of variables, while the center system, $\mathbf{z_2}$ may have different internal dynamics, given by $\widetilde{\mathbf{F}}$, and different coupling function $\widetilde{\mathbf{G}}$. The delay in the coupling terms is fixed and given by τ, and the strength of coupling from the center to the outer identical subsystems by δ_1, while from the outer to the center by δ_2. Due to internal symmetry of the system, there exists an identical solution for the outside subsystems, $\mathbf{z_1}(t) = \mathbf{z_3}(t) = \phi(t)$. If the conditional Lyapunov exponents calculated with respect to perturbation out of the synchronization manifold, $\mathbf{z_1}(t) = \mathbf{z_3}(t)$ are all negative, then the two outer subsystems are synchronized. This type of behavior where the two systems show identical dynamics, even chaotic ones, is called complete synchronization. Figure 2 shows an example of complete synchronization in the case of a mutually coupled three laser system using the model in Sec. III [10]. Calculating Lyapunov exponents is in general complicated due to the presence of time delays in the equations. The coupling term containing delays, however, drops out if Eqs. (1) and (3) are linearized about the synchronous state: $\mathbf{z_1}(t) = \mathbf{z_3}(t)$. To study the stability of the symmetric solution, we introduce new variables, $\triangle\mathbf{z_1}(t) = \mathbf{z_1}(t) - \phi(t)$ and $\triangle\mathbf{z_3}(t) = \mathbf{z_3}(t) - \phi(t)$. Linearizing transverse to the synchronization manifold, we have

$$\frac{d \triangle \mathbf{z_1}(t)}{dt} = \mathbf{J} \cdot \triangle \mathbf{z_1}(t), \qquad (4)$$

$$\frac{d \triangle \mathbf{z_3}(t)}{dt} = \mathbf{J} \cdot \triangle \mathbf{z_3}(t), \qquad (5)$$

where $\mathbf{J}$ is the $\mathbf{M} \times \mathbf{M}$ Jacobian matrix of partial derivatives evaluated at $\phi(t)$,

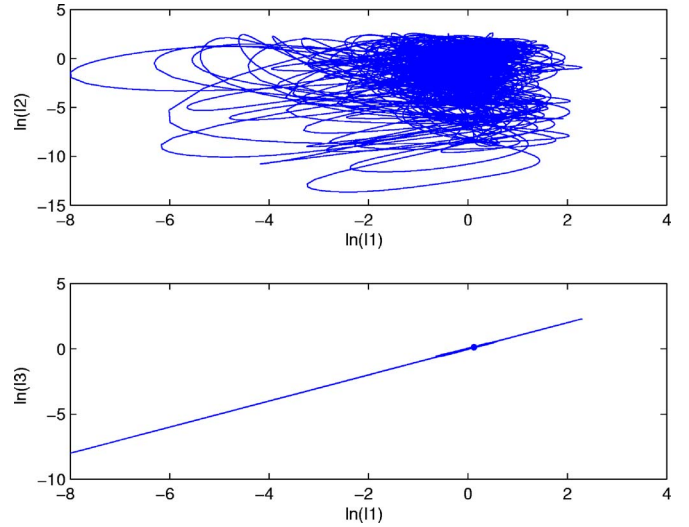$$\mathbf{J} = \frac{\partial \mathbf{F}(\phi(t))}{\partial \mathbf{z}}. \qquad (6)$$

FIG. 2. (Color online) Top: Intensity of laser 1 vs laser 2. Bottom: Laser 1 vs laser 3. Straight line indicates complete synchronization of outer lasers. $\tau = 30$, $\epsilon = \sqrt{0.001}$, $\delta_1 = \delta_2 = 6.5\epsilon$.

Here $\phi(t)$ is the synchronous state that is determined by the dynamics of Eqs. (1)–(3) and the initial conditions defined on $[-\tau, 0]$. Although the time delays and dependence on $\mathbf{z_2}$ drop out of Eqs. (4) and (5), they are involved implicitly in determining $\phi(t)$, the synchronization manifold. Equations (4) and (5) are M-dimensional and therefore have M transverse Lyapunov exponents. The largest of them determines the stability of the transverse perturbation. So that the synchronized state $\mathbf{z_1}(t) = \mathbf{z_3}(t) = \phi(t)$ is asymptotically stable, if $\triangle\mathbf{z_{1,3}} \rightarrow 0$ as $t \rightarrow \infty$ or if all of the Lyapunov exponents in the linearized equations are negative.

For $\delta_2 = 0$, in Eq. (2), the dynamics of $\mathbf{z_{1,3}}$ become that of a driven system, with $\mathbf{z_2}$ acting as the driver. Then, the synchronized dynamics correspond to generalized synchronization [11] whereby the driven subsystem becomes a function of the driver, $\phi(t) = \mathbf{f}(\mathbf{\Phi})$. Here, $\mathbf{\Phi}$ are the dynamics of the driver obtained by integrating Eq. (2), with $\delta_2 = 0$, $d\mathbf{\Phi}(t)/dt = \widetilde{F}(\mathbf{\Phi}(t))$.

While the exact form of the function between the driver and the driven systems can be rather complicated and difficult to obtain, its existence can be inferred from the synchronization of identical systems when started from different initial conditions and being exposed to the same drive. This method of detecting generalized synchronization using identical driven systems is known as the auxiliary systems approach [12]. In order for the driven subsystems, $\mathbf{z_{1,3}}$ to become synchronized, their dependence on initial conditions must "wash out" as a function of time. This is due to the fact that dependence on initial conditions prevents synchronization by making the dynamics of $\mathbf{z_1}$ different from the dynamics of $\mathbf{z_3}$. This "washing out" of initial conditions is provided by the dissipation in the system, which must therefore be either present in the coupling term, or in the uncoupled dynamics of the system itself. This can be seen by taking the sum of Lyapunov exponents, for Eqs. (4) and (5), which is related to contraction or expansion of the phase-space volume of the dynamics transverse to the synchronization manifold [13],

$$\sum_{j=1}^{M} \lambda_j = \lim_{t \to \infty} \frac{1}{t} \ln \left| \det(\mathbf{\Psi}(\triangle \mathbf{z_{1,3}})(t)) \right|, \qquad (7)$$

where $\mathbf{\Psi}$ is the fundamental matrix solution to Eqs. (4) and (5) From Eq. (7), negative Lyapunovs exponents sum corresponds to a contraction of phase space of $\triangle \mathbf{z_{1,3}}$ dynamics as a function of time. In the case of synchronization in driven identical systems, all Lyapunov exponents transverse to the synchronous solution must be negative. Thus there is a contraction of phase space to a single trajectory that is a function of the dynamics of the driver, $\mathbf{\Phi}$. This shrinking of phase space is either caused by dissipative coupling or dissipation in the driven systems themselves. The effect of dissipation on synchronization can be illustrated using the case of a simple driven system,

$$\frac{dx}{dt} = - \epsilon x + \Phi(t), \quad \frac{dy}{dt} = - \epsilon y + \Phi(t) \qquad (8)$$

taking the difference between the two variables, $\triangle = x - y$, we get, $d\triangle/dt = -\epsilon \triangle$, where $\Phi(t)$ in Eq. (8) is the chaotic signal provided by the driver (it can also be a noisy signal) [3,14]. Thus the difference in initial conditions between x and y decreases at the rate of dissipation, ϵ, leading to synchronization for large times.

In mutually coupled systems, $\delta_2 \neq 0$, the dynamics of $\mathbf{z_2}$ are affected by $\mathbf{z_1}$ and $\mathbf{z_3}$. In this case, the synchronized state, $\phi(t)$, may depend on the initial conditions of all of the three subsystems, $\{\mathbf{z_1}, \mathbf{z_2}, \mathbf{z_3}\}$, so that $\phi(t)$ cannot be the result of generalized synchronization, in a strict sense. However, it takes a time interval of 2τ for any change in the dynamics of systems $\mathbf{z_{1,3}}$ to affect the motion of these systems via mutual coupling. During this time interval of length, 2τ, $\mathbf{z_{1,3}}$ can be viewed driven by $\mathbf{z_2}$, since the signal $\mathbf{z_{1,3}}$ receives during that time interval is not affected by its dynamics on that interval. Therefore, we examine the dynamics in a time period on the order of the delay time, τ.

The initial dynamics of $\mathbf{z_{1,3}}(t)$ on the time interval $t_0 \leq t < t_0 + \tau$, affect the dynamics of $\mathbf{z_{1,3}}(t)$ on the time interval $t_0 + 2\tau \leq t < t_0 + 3\tau$ via the mutual coupling term, $\mathbf{G}(\mathbf{z_2}(t - \tau))$ [see Eqs. (1) and (3)]. For chaotic systems, the trajectories are not correlated in time, and we assume

$$\langle \mathbf{z_i}(t) \cdot \mathbf{z_i}(t - t_0) \rangle \approx 0 \qquad (9)$$

for $i = \{1, 2, 3\}$ and t_0 significantly longer than the average period of oscillation. The above equation is true, in general, for nonperiodic oscillations. This can be seen by expanding the signal in a Fourier series, $z_i(t) = \sum_{n=-\infty}^{\infty} A_n \cos(nt + \phi)$. Then, if there is a significant periodic component in $z_i(t)$ of amplitude A_n, Eq. (9) will be proportional to A_n^2 whenever t_0 is a multiple of $2\pi/n$. Thus, for example, Eq. (9) may not hold if the amplitude of an optoelectronically coupled laser (to be discussed in the following section) is too close to the threshold, where the behavior can be approximated as coupled linear oscillators [see Eqs. (10)–(12)], leading to a significant regular oscillatory component in the signal. In this case, there may be a resonant interaction between the lasers, which is sensitive to the specific value of the coupling delay, τ. For chaotic dynamics, we can substitute $t_0 = 2\tau$ into Eq.

(9), where τ is the delay, to see that there is no significant correlation between the dynamics on that time scale, so that over the round-trip time of 2τ, the identical subsystems $\mathbf{z_{1,3}}$ can be viewed as driven by some uncorrelated chaotic signal coming from $\mathbf{z_2}$. This assumption of a driver is only strictly valid on the time interval within the round-trip time, since for longer time intervals the initial conditions of the outer lasers at the beginning of the interval will affect their dynamics, via the middle system, at a later time within the interval.

By perturbing the dynamics of the outer subsystems from the synchronized state, it can be shown that complete synchronization of the end subsystems in the presence of long delays is similar to generalized synchronization, where the middle subsystem acts as the driver for the outer ones. After the symmetric subsystems synchronize, $\mathbf{z_{1,3}}(t) = \phi(t)$, one of them can be suddenly perturbed from its symmetric state to an arbitrary position in phase space at some $t = t_1$. In that case, the perturbed system, as well as the unperturbed one, will receive the exact same signal from $\mathbf{z_2}$ as before, for $t < t_1 + 2\tau$. If the systems synchronize again at some point during $t < t_1 + 2\tau$, we will again have $\mathbf{z_{1,3}}(t) = \phi(t)$, where $\phi(t)$ has not been affected by the perturbation during that time interval. Thus the synchronized state, $\phi(t) = \mathbf{z_{1,3}}(t)$ is clearly independent of perturbations of subsystems $\mathbf{z_{1,3}}$ and must therefore be some function of the middle subsystem, $\mathbf{z_2}$. This however is the same as what happens in generalized synchronization, with the difference that the trajectory of $\mathbf{z_2}$ itself may be affected by the initial starting conditions of the symmetric subsystems. The following section uses a system of semiconductor lasers as an example for application of these ideas to understand complete synchronization of the end lasers in a three laser system.

III. SYNCHRONIZATION OF SEMICONDUCTOR LASERS WITH DELAYS

The discussion of the preceding section can be applied to the study of synchronization of a three laser system with delays. A schematic diagram is shown in Fig. 1, where the outer lasers are identical, while the middle laser is detuned.

The scaled equations of coupled semiconductor lasers have the following form [15,16]:

$$\frac{dy_1}{dt} = x_1(1 + y_1),$$

$$\frac{dx_1}{dt} = - y_1 - \epsilon x_1(a_1 + b_1 y_1) + \delta_2 y_2(t - \tau), \qquad (10)$$

$$\frac{dy_2}{dt} = \beta x_2(1 + y_2),$$

$$\frac{dx_2}{dt} = \beta[- y_2 - \epsilon \beta x_2(a_2 + b_2 y_2)] + \delta_1[y_1(t - \tau) + y_3(t - \tau)], \qquad (11)$$

$$\frac{dy_3}{dt} = x_3(1 + y_3),$$

$$\frac{dx_3}{dt} = -y_3 - \epsilon x_3(a_1 + by_1) + \delta_2 y_2(t - \tau). \quad (12)$$

Equations (10)–(12) have the same form as Eqs. (1)–(3) with $\mathbf{z_i} = \{y_i, x_i\}$, where $i = 1, 3$ and $i = 2$ for the outer and middle lasers, respectively. Variables y_i and x_i denote scaled intensity and inversion of the ith laser, $\{a_1, a_2, b_1, b_2\}$ are loss terms, and ϵ is the dissipation. (See Ref. [17] for details of the derivation from the original physical model.) The above equations are coupled via laser intensities, y_i, using optoelectronic incoherent coupling that does not contain electric field phase information, unlike the coherent coupling by Fischer *et al.* [9]. Previously, the dynamics of two electronically coupled lasers have been explored [18], showing lag synchronization for the case of two lasers, and isochronal synchronization if feedback is added [19]. The above equations are scaled so that the relaxation frequency is equal to 1. In the typical experimental setup, the relaxation oscillations are on the order of 2–3 ns. Since the delay time is scaled by the relaxation frequency, and is at least an order of magnitude higher (for long delays), a typical delay time used in simulations could be about $\tau = 60$, which corresponds to about 20–30 ns.

In the absence of dissipation, the uncoupled system, $\delta_1 = \delta_2 = 0$, is a nonlinear conservative system, with behavior similar to a simple harmonic oscillator for small amplitudes, and becoming more pulselike at high amplitudes [17]. Dissipation, however, leads to energy loss, so that in the absence of coupling between lasers, the dynamics would settle into a steady state. Thus mutual coupling acts like a drive by pumping energy into the system. For most cases, it can be assumed that dissipation is small, $\epsilon \ll 1$. Detuning of the middle laser from the outer ones is given by β.

The system described by Eqs. (10)–(12) shows complete synchronization of outer lasers over a whole range of parameters. Figure 2 shows that while the outer lasers can become completely synchronized, there may be no apparent correlation between the middle and the outer lasers. Since the outer lasers are identical, there is a solution of Eqs. (10)–(12) where $y_1 = y_3 = Y(t)$ and $x_1 = x_3 = X(t)$. In this case, Eqs. (10)–(12) reduce to four differential equations. The solution $y_1 = y_3$ and $x_1 = x_3$ is stable if the Lyapunov exponents transverse to the synchronization manifold are negative. To investigate the stability of the synchronized state we linearize about the synchronous solution $\phi(t) = \{X(t), Y(t)\}$. Applying Eqs. (4)–(6) to Eqs. (10) and (12), we obtain

$$\begin{pmatrix} \triangle \dot{x}_{1,3} \\ \triangle \dot{y}_{1,3} \end{pmatrix} = \begin{pmatrix} -\epsilon[a_1 + b_1 Y(t)] & -1 \\ 1 + Y(t) & X(t) \end{pmatrix} \begin{pmatrix} \triangle x(t)_{1,3} \\ \triangle y(t)_{1,3} \end{pmatrix}, \quad (13)$$

where $\{\triangle x(t)_{1,3}, \triangle y(t)_{1,3}\}$ are perturbations of outer oscillators from the synchronous state $\{X(t), Y(t)\}$. This synchronous state is obtained by starting the outer oscillators from the same initial conditions and perturbing the system at some time, $t = t_1$. The perturbation will not affect the coefficient matrix in Eq. (13) until $t \geq t_1 + 2\tau$. So that in the time interval

of 2τ the dynamics off the synchronization manifold can be viewed as driven by an uncorrelated chaotic signal $\{X(t), Y(t)\}$. We can now apply Abel's formula [13] to Eq. (13), which relates the Wronskian of the linearized system to the trace of the matrix [20]. Dropping the subscripts on linearized variables, we get,

$$W(t) = \det \begin{vmatrix} \triangle x & \triangle y \\ \triangle \dot{x} & \triangle \dot{y} \end{vmatrix} = \exp\left(\int_{t_1}^{t} \{X(s) - \epsilon[a_1 + b_1 Y(s)]\}ds \right).$$
$$(14)$$

The Wronskian gives the phase-space volume dynamics of the system $\{\triangle x(t), \triangle y(t)\}$. Equation (14) is valid over the integration interval of 2 times the delay, $t_1 < t < t_1 + 2\tau$. This is due to the fact that it takes a time interval of 2τ for a perturbation in the outer laser to affect its dynamics via mutual coupling from the middle laser. Thus, during the time interval of 2τ, the perturbed system acts like a driven system in that its dynamics do not affect the signal it receives, and therefore do not change the synchronized state, $\{X, Y\}$, making it independent of $\{\triangle x, \triangle y\}$ dynamics over the integration interval.

Since the variable $Y(t)$ is the scaled intensity of the laser, from Eqs. (10)–(12), its minimum possible value is -1. Thus for $a_1 > b_1$ (a typical case), the contribution of the dissipation term to the Wronskian is always negative. The variable $X(t)$, on the other hand, is symmetric about zero, and thus averages out to zero when integrated over a single period of oscillation. It follows that if the integral in Eq. (14) is taken just over a single oscillation of the laser, we obtain

$$\int_{t_1}^{t_1+T} \{X(s) - \epsilon[a_1 + b_1 Y(s)]\}ds = -\epsilon(a_1 + b_1 \bar{Y})T < 0,$$
$$(15)$$

where T is the period of a single oscillation, and $\bar{Y}$ is the average of Y over a single period (unlike X, the Y variable is not symmetric about zero, which can readily be seen in the pulselike fluctuations of lasers at high intensities). It follows, that $X(s)$ in Eq. (14) averages out to zero if the integral is done over many periods of oscillation, while the dissipation term, multiplied by ϵ, provides a continuous negative component. If that continuous negative component builds up sufficiently over the integration interval to overcome any fluctuations in $X(s)$, we then have a continuous shrinking of the phase space of perturbed dynamics, indicating synchronization. Integrating the exponential term in Eq. (14) over many oscillations and using Eq. (15), we obtain

$$\int_{t_1}^{t} \{X(s) - \epsilon[a_1 + b_1 Y(s)]\}ds \approx -\epsilon(a_1 + b_1 \bar{Y})(t - t_1)$$

$$+ \int_{t_1+nT}^{t} X(s)ds, \quad (16)$$

where $t - t_1$ is the total integration interval, $\bar{Y}$ is the average value of intensity over that interval, and n on the integration limits is the total integer amount of full oscillations that fit into the integration period, $t - t_1 - nT < T$. Here, T is the av-

erage period of oscillation over the integration interval. Thus the integral of $X(s)$ on the right-hand side is only over a single uncompleted oscillation. This integral, however, may still be significant compared to the ϵ term, since its fluctuations are comparable to $X(s)$, because the integration period, T, is of order unity (due to scaling in the equations), while ϵ multiplying the other term is small. It follows that sufficiently long integration times, $t-t_1$, are required in order for the dissipation term to dominate. We can now set an upper bound for the integral on the right-hand side of Eq. (16),

$$\int_{t_1+nT}^{t} X(s)ds < \pi |X(t)|_{\text{max}}, \tag{17}$$

where $|X(t)|_{\text{max}}$ is the maximum fluctuation of inversion over the interval of 2 times the delay time. The above bound may not be valid at energies far above the threshold, when the laser behavior becomes pulselike with a period that is significantly longer than the scaled relaxation period of 2π. This may be another reason why there is a loss of synchronization at higher coupling strengths, which lead to higher amplitudes of oscillation, with lower frequencies. Requiring Eq. (16) to be less than zero and using Eq. (17), we can now obtain a bound above which the dynamics tend toward the synchronization manifold over the interval of 2 times the delay time,

$$\frac{2}{\pi}\tau\epsilon(a_1 + b_1\bar{Y}) > |X(t)|_{\text{max}}, \tag{18}$$

where we have used $2\tau = t - t_1$ for the integration interval. The above inequality insures the right-hand side of Eq. (16) is negative over the interval of 2 times the delay time. This in turn ensures the shrinking volume of transverse phase-space dynamics given by Eq. (14).

Equation (18) gives a condition for the phase-space volume of transverse dynamics to contract over the interval of 2 times the delay. For sufficiently long delays, where $\tau\epsilon(a_1+b_1\bar{Y}) \gg |X(t)|_{\text{max}}$, Eq. (14) can be approximated as

$$\ln[W(t)] = \ln|\bigtriangleup x\dot{\bigtriangleup} y - \bigtriangleup y\dot{\bigtriangleup} x| \approx -\int_{t_1}^{t} \epsilon[a_1 + b_1 Y(s)]ds. \tag{19}$$

where a natural log of $W(t)$ was taken. The above equation is a monotonically decreasing function of t. This means that the phase-space volume of the system perturbed from the synchronization manifold contracts as a function of time. Applying Eq. (7) to Eq. (19), where $\det(\boldsymbol{\Psi}(\triangle\mathbf{z_{1,3}})) = W(t)$, the sum of transverse Lyapunov exponents for the dynamics off the synchronization manifold described by Eq. (13) can now be approximated as

$$\lambda_1 + \lambda_2 \approx -\epsilon(a_1 + b_1\bar{Y}). \tag{20}$$

Equation (20) indicates that for sufficiently long integration times (requiring sufficiently long delays), the sum of the Lyapunov exponents should be negative, indicating the shrinking of phase-space volume of dynamics transverse to the synchronization manifold.
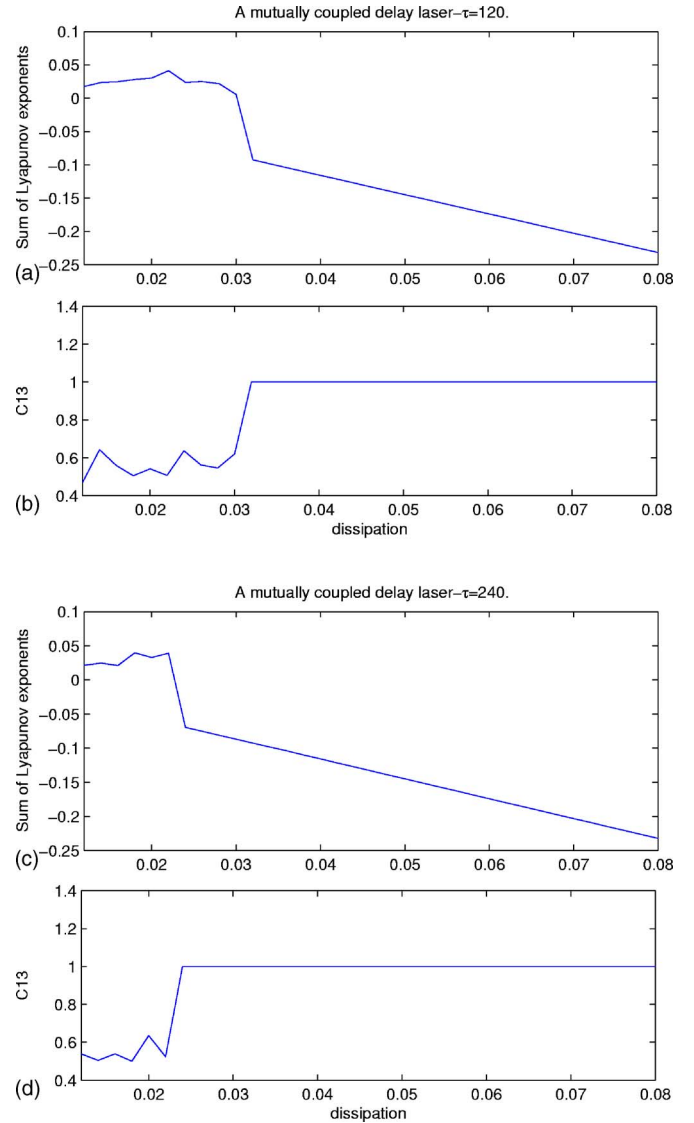

FIG. 3. (Color online) (a) Sum of Lyapunov exponents as a function of dissipation, ϵ, for $\tau = 120$. (b) Corresponding correlations between outer lasers, $\tau = 120$. (c) Sum of Lyapunov exponents vs ϵ, for $\tau = 240$. (d) Corresponding correlations between outer lasers, $\tau = 240$. In all cases, $a_1 = a_2 = 2$, $b_1 = b_2 = 1$, $\delta_1 = \delta_2 = 0.2$, $\beta = 0.5$.

Figure 3 shows numerically computed sum of Lyapunov exponents, and corresponding correlations of the outer lasers, as a function of dissipation, ϵ, for two values of the delay, $\tau = 120$ and $\tau = 240$. The fluctuations in the sum of Lyapunov exponents correspond well to the fluctuations in the correlation function of the outer lasers, with desynchronization when the Lyapunov sum increases above zero. As might be expected from Eq. (18), longer delays mean synchronization at lower values of dissipation, since the dissipation term in the exponential in Eq. (14) dominates for sufficiently long delays. Increasing τ by a factor of 2, however, does not decrease the bifurcation value of ϵ for the onset of synchronization by a factor of 2, as might be expected from Eq. (18). This is probably due to the decrease in fluctuations, $|X(t)|_{\text{max}}$, as the dissipation in the system is increased, leading to synchronization at a lower value of ϵ than might otherwise be

expected. After Eq. (18) is satisfied, resulting in the onset of synchronization, the sum of Lyapunov exponents has a negative linear dependence, given by Eq. (20). This is in agreement with Fig. 3 which shows this negative linear dependence of Lyapunov sum on dissipation, with a slope of around −2.6, a reasonable value for the parameters used of $a_1=2$, $b_1=1$ and intensity, $\bar{Y} \sim 1$. In general, the average intensity of the dynamics, $\bar{Y}$, depends on the coupling strengths, δ_1, and δ_2.

While Eqs. (19) and (20) predict the shrinking of phase space for the dynamics transverse to the synchronization manifold, to guarantee stability both Lyapunov exponents must be negative, or the solution will blow up along the unstable direction. To find out whether the synchronous state is stable, consider again the Wronskian, $W(t) = \triangle x \dot{\triangle} y - \triangle y \dot{\triangle} x$. Substituting for $\dot{\triangle} x$ and $\dot{\triangle} y$ from Eq. (13), we get

$$W(t) = [1 + Y(t)](\triangle x)^2 + (\triangle y)^2 + \{\epsilon[a_1 + b_1 Y(t)] \\ + X(t)\} \triangle x \triangle y. \quad (21)$$

In Eq. (19), $W(t)$ is a monotonically decreasing function of time, with $W(t) \to 0$ as $t \to \infty$. Therefore both $\triangle x$, $\triangle y \to 0$ as $t \to \infty$, due to the presence of terms quadratic in $\triangle x$ and $\triangle y$ in Eq. (21). It follows that for sufficiently long delays in the system, the synchronized state is stable, and therefore all the Lyapunov exponents transverse to the synchronization manifold are negative. The stability of synchronized state is due to cross-terms in the matrix in Eq. (13), which come from rotation, leading to the nonlinear exchange of energy between inversion and intensity of the laser. This rotation introduces terms quadratic in $\triangle x$ and $\triangle y$ into Eq. (21) and leads to the spiraling of the phase-space volume towards zero, rather that blowing up along one direction, while shrinking along another.

Figure 4(a) shows the sum of Lyapunov exponents as a function of delay. The Lyapunov exponents are negative for all $\tau > 170$ (corresponding to about 60 ns) resulting in complete synchronization of the outer lasers, as shown in Fig. 4(b). At the same time, the outer lasers are not synchronized with the center one, Fig. 4(c). The fluctuations in correlations of the outer lasers match well the fluctuations in the Lyapunov sum, with correlations increasing whenever the Lyapunov sum decreases. Figure 4 agrees well with the above analysis, since sufficiently long delays [see Eq. (18)] are needed for the Lyapunov exponents to become negative, leading to synchronization. After the onset of synchronization, Eq. (20) becomes valid, so that the Lyapunov sum becomes independent of delays. This is confirmed by the straight horizontal line in the figure, after the outer lasers synchronize. The degree of synchronization is given by the correlations function.

From Eq. (20), the negative Lyapunov exponents, leading to stability of synchronous state, are the result of dissipation, ϵ, in the end lasers. This is to be expected since mutual coupling pumps energy into the system, as can be seen in Eqs. (10)–(12). Therefore, some dissipation in the outer subsystems themselves is essential in order to "wash out" their
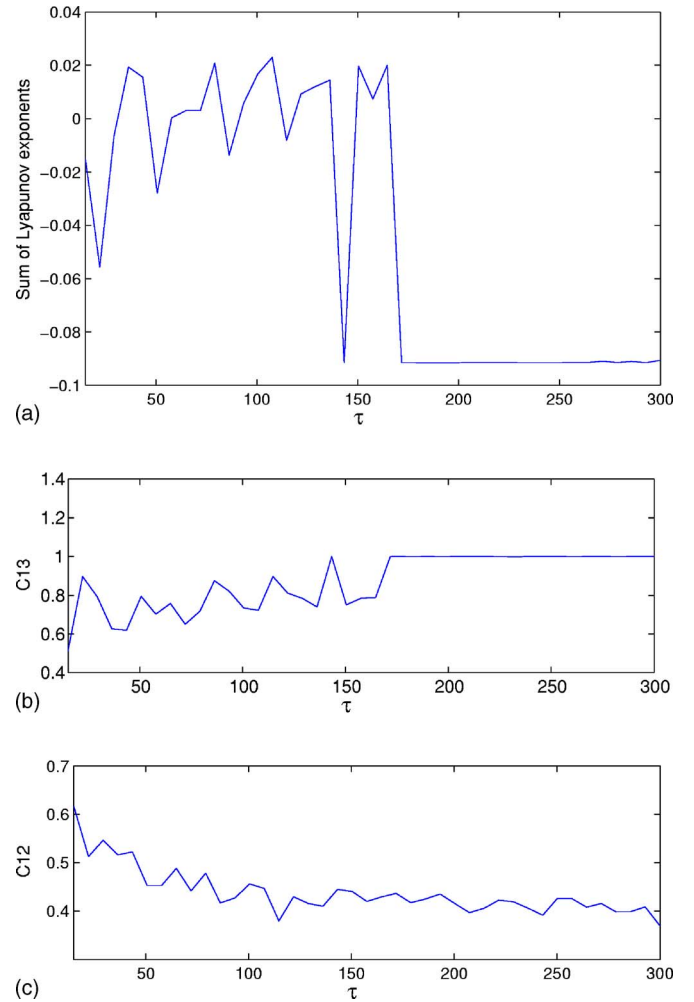

FIG. 4. (Color online) (a) Numerically computed sum of Lyapunov exponents as a function of delay, τ. (b) Corresponding correlations of outer lasers. (c) Correlations of the middle and outer lasers, shifted by the delay time to maximize correlations. $\epsilon = \sqrt{0.001}$, $\delta_1 = \delta_2 = 7.5\epsilon$, $\beta = 0.5$.

dependence on initial conditions and make them a function of the dynamics of the middle laser, as would be required in the case on complete synchronization.

The amplitude of laser oscillations depends on the coupling strengths, δ_1 and δ_2, as well as the dissipation. It was shown [15] that in the case of a two laser system there is a bifurcation value for the onset of oscillations that is a function of product of the coupling strengths, $\delta_1 \delta_2$. Increasing the coupling strengths increases the role of nonlinearities in the system and the intensity of laser oscillations. Thus for low values of the coupling strengths, the dynamics given by Eqs. (10)–(12) are more regular. At low intensities, the dynamics of individual lasers are close to that of a simple harmonic oscillator, as can be verified by substituting low values of $\{x_1, y_1\}$ into Eq. (10), for example. In order for Eq. (9) to be valid, the dynamics must be uncorrelated over the time interval of the delay. Thus the equations derived in this section are valid for chaotic regime which requires sufficiently high product of coupling strengths, $\delta_1 \delta_2$. In this case, it can be assumed that the outer lasers are driven by an uncorrelated
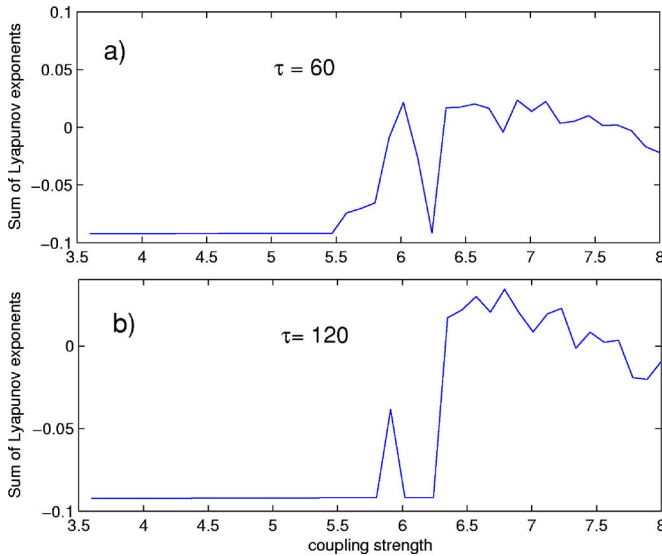
FIG. 5. (Color online) (a) Sum of Lyapunov exponents as a function of coupling strength, $\delta_1 = \delta_2$, for $\tau = 60$. (b) $\tau = 120$. $\epsilon = \sqrt{0.001}$, $\beta = 0.5$.

signal from the middle laser over the time interval of 2τ. Any synchronization on that interval would then be analogous to generalized synchronization that occurs in a unidirectional system, with the middle laser acting as the driver for the outer ones. Since increased coupling pumps more energy into the system, thereby increasing the effect of nonlinearities, the Lyapunov exponents may increase above zero, leading to desynchronization of the outer lasers. In this case, longer delays in coupling may be required in order for the outer lasers to synchronize. This effect is illustrated in Fig. 5, which shows the sum of Lyapunov exponents as a function of coupling strengths for two different delays, $\tau = 60$ and $\tau = 120$. There is an abrupt increase in Lyapunov exponents above zero, due to increased nonlinearity, as the coupling strength is increased. Increasing the delay however to $\tau = 120$ leads to synchronization for a greater range of coupling strengths, as compared to $\tau = 60$. The corresponding correlations as a function of coupling strengths are shown in Fig. 6. Desynchronization at higher coupling strengths, and the synchronizing effect of increased delays is in agreement with Eq. (18). Since higher coupling strengths lead to greater fluctuations in $X(t)$, as well as a slower period of oscillation, longer integration times are required in order for Eqs. (19) and (20) to be valid, leading to synchronization at longer delays, τ.

Figures 3–6 show that Eq. (20) correctly predicts the independence of Lyapunov sum on delays and coupling strengths and a negative linear dependence on dissipation, once synchronization sets in. Synchronization, on the other hand, occurs once the condition expressed in Eq. (18) is satisfied, leading to the continuous shrinking of the phase-space dynamics transverse to the synchronization manifold.

IV. CONCLUSION

Ideas from generalized synchronization were used to explain complete chaotic synchronization of mutually coupled
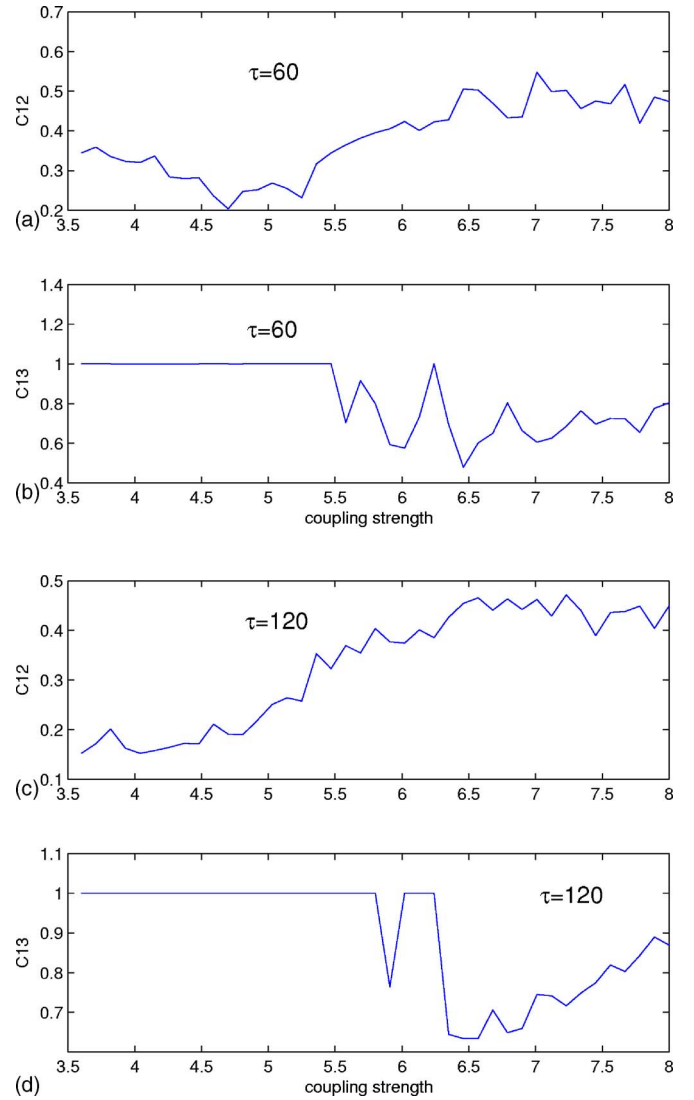

FIG. 6. (Color online) Correlations corresponding to Fig. 5. (a) Correlation between the middle and one of the outer lasers, $\tau = 60$. (b) Correlations of outer lasers, $\tau = 60$. (c) Correlation between the middle and one of the outer lasers, $\tau = 120$. (d) Correlations of outer lasers, $\tau = 120$. Outer lasers synchronize for greater range of coupling strength as the delay is increased. The middle and the outer lasers show little correlation for all values of the coupling strengths.

systems in the presence of long delays. Since identical outer subsystems synchronize due to a common input from the middle subsystem, complete synchronization is similar to the one occurring in the auxiliary system setup, with the exception that all subsystems are mutually coupled. This leads to the dependence of common input to the outer subsystems on history of the dynamics. Complete chaotic synchronization is the result of the outer systems becoming a function of the middle one, as would happen in the case of generalized synchronization.

Due to the symmetry of the outer subsystems, the dynamics linearized about the synchronization manifold are independent of explicit coupling. Transverse Lyapunov exponents can then be calculated to determine the stability of the synchronous state. Since over the time scale of 2 times the

delay interval, the outer subsystems can be viewed as driven by a common chaotic signal from the middle subsystem, the analysis is considerably simplified, allowing for calculation of phase-space volume dynamics transverse to the synchronization manifold. The transverse phase-space volume dynamics were analyzed for the case of three mutually coupled semiconductor lasers. It was found that for sufficiently long delays, the synchronized state is stable. The sum of Lyapunov exponents transverse to the synchronization manifold was found analytically and shown to have a negative linear dependence on dissipation, in good agreement with numerical calculations. This also confirmed the intuition that synchronization is the result of dissipation, ϵ in local dynamics of the lasers themselves, since the coupling between lasers is not dissipative. The analysis also explains the effect of various parameters on synchronization, such as cou-

pling strengths, delay time, and dissipation, and is supported by numerical simulations over a range of parameter values. Thus, it was shown analytically and confirmed numerically that after the onset of synchronization, the stability of the synchronous state (as given by Lyapunov exponents) depends linearly on dissipation, but is independent of the delay time and coupling strength.

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